

Count of eigenvalues in the generalized eigenvalue problem

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Abstract

We address the count of isolated and embedded eigenvalues in a generalized eigenvalue problem defined by two self-adjoint operators with a positive essential spectrum and a finite number of isolated eigenvalues. The generalized eigenvalue problem determines spectral stability of nonlinear waves in a Hamiltonian dynamical system. The theory is based on the Pontryagin's Invariant Subspace theorem in an indefinite inner product space but it extends beyond the scope of earlier papers of Pontryagin, Krein, Grillakis, and others. Our main results are (i) the number of unstable and potentially unstable eigenvalues *equals* the number of negative eigenvalues of the self-adjoint operators, (ii) the total number of isolated eigenvalues of the generalized eigenvalue problem is *bounded from above* by the total number of isolated eigenvalues of the self-adjoint operators, and (iii) the quadratic form defined by the indefinite inner product is strictly positive on the subspace related to the absolutely continuous part of the spectrum of the generalized eigenvalue problem. Applications to solitons and vortices of the nonlinear Schrödinger equations and solitons of the Korteweg–De Vries equations are developed from the general theory.

Keywords: generalized eigenvalue problem, discrete and continuous spectrum, indefinite metric, invariant subspaces, isolated eigenvalues, Krein signature

1 Introduction

Stability of equilibrium points in a Hamiltonian system of finitely many interacting particles is defined by the eigenvalues of the generalized eigenvalue problem [GK02],

$$A\mathbf{u} = \gamma K\mathbf{u}, \quad \mathbf{u} \in \mathbb{R}^n, \quad (1.1)$$

where A and K are symmetric matrices in $\mathbb{R}^{n \times n}$ which define the quadratic forms for potential and kinetic energies, respectively. The eigenvalue γ corresponds to the normal frequency $\lambda = i\omega$ of the normal mode of the linearized Hamiltonian system near the equilibrium point, such that $\gamma = -\lambda^2 = \omega^2$. The linearized Hamiltonian system is said to have an unstable eigenvalue γ if $\gamma < 0$ or $\text{Im}(\gamma) \neq 0$. Otherwise, the system is weakly spectrally stable. Moreover, the equilibrium point is a minimizer of the Hamiltonian if all eigenvalues γ are positive and semi-simple and the quadratic forms for potential and kinetic energies evaluated at eigenvectors of (1.1) are strictly positive.

When the matrix K is positive definite, all eigenvalues γ are real and semi-simple (that is the geometric and algebraic multiplicities coincide). By the Sylvester's Inertia Law theorem [G61], the numbers of positive, zero and negative eigenvalues of the generalized eigenvalue problem (1.1) *equal to* the numbers of positive, zero and negative eigenvalues of the matrix A . When K is not positive definite, a complete classification of eigenvalues γ in terms of real eigenvalues of A and K has been developed with the use of the Pontryagin's Invariant Subspace theorem [P44], which generalizes the Sylvester's Inertia Law theorem.

We are concerned with spectral stability of spatially localized solutions in a Hamiltonian infinite-dimensional dynamical system. In many problems, a linearization of the nonlinear system at the spatially localized solution results in the generalized eigenvalue problem of the form (1.1) but A and K^{-1} are now self-adjoint operators on a complete infinite-dimensional metric space. There has been recently a rapidly growing sequence of publications on mathematical analysis of the spectral stability problem in the context of nonlinear Schrödinger equations and other nonlinear evolution equations [CPV05, GKP04, KKS04, KP05, KS05, P05]. Besides predictions of spectral stability or instability of spatially localized solutions in Hamiltonian dynamical systems, linearized Hamiltonian systems are important in analysis of orbital stability [GSS87, GSS90, CP03], asymptotic stability [P04, RSS05, C03], stable manifolds [CCP05, S05], and blow-up of solutions in nonlinear equations [P01].

It is the purpose of this article to develop analysis of the generalized eigenvalue problem in infinite dimensions by using the Pontryagin space decomposition [P44]. The theory of Pontryagin spaces was developed by M.D. Krein and his students (see books [AI86, GK69, IKL82]) and partly used in the context of spectral stability of solitary waves by MacKay [M87], Grillakis [G90], and Buslaev & Perelman [BP93] (see also a recent application in [GKP04]). We shall give an elegant geometric proof of the Pontryagin's Invariant Subspace theorem. An application of the theorem recovers the main results obtained in [CPV05, KKS04, P05]. Moreover, we obtain a new *inequality* on the number of positive eigenvalues of

the linearized Hamiltonian that extends the count of eigenvalues of the generalized eigenvalue problem.

The structure of the paper is as follows. Main formalism of the generalized eigenvalue problem is described in Section 2. The Pontryagin's Invariant Subspace theorem is proved in Section 3. Main results on eigenvalues of the generalized eigenvalue problem are formulated and proved in Section 4. Section 5 contains applications of the main results to solitons and vortices of the nonlinear Schrodinger equations and solitons of the Korteweg–De Vries equations.

2 Formalism

Let L_+ and L_- be two self-adjoint operators defined on the Hilbert space \mathcal{X} with the inner product (\cdot, \cdot) . Our main assumptions are listed below.

P1 The essential spectrum of L_\pm in \mathcal{X} includes the absolute continuous part $\sigma_c(L_\pm)$ bounded from below by $\omega_+ \geq 0$ and $\omega_- > 0$ and finitely many embedded eigenvalues of finite multiplicities.

P2 The discrete spectrum of L_\pm in \mathcal{X} includes finitely many isolated eigenvalues of finite multiplicities with $p(L_\pm)$ positive, $z(L_\pm)$ zero, and $n(L_\pm)$ negative eigenvalues.

We shall consider the linearized Hamiltonian problem defined by the self-adjoint operators L_\pm in \mathcal{X} ,

$$L_+u = -\lambda w, \quad L_-w = \lambda u, \quad (2.1)$$

where $\lambda \in \mathbb{C}$ and $(u, w) \in \mathcal{X} \times \mathcal{X}$. By assumption P1, the kernel of L_- is isolated from the essential spectrum. Let \mathcal{H} be the constrained Hilbert space,

$$\mathcal{H} = \{u \in \mathcal{X} : u \perp \ker(L_-)\}, \quad (2.2)$$

and let \mathcal{P} be the orthogonal projection from \mathcal{X} to \mathcal{H} . The linearized Hamiltonian problem (2.1) for non-zero eigenvalues $\lambda \neq 0$ is rewritten as the generalized eigenvalue problem

$$Au = \gamma Ku, \quad u \in \mathcal{H}, \quad (2.3)$$

where $A = \mathcal{P}L_+\mathcal{P}$, $K = \mathcal{P}L_-\mathcal{P}$, and $\gamma = -\lambda^2$. We note that K is a bounded invertible self-adjoint operator on \mathcal{H} , while properties of A follow from those of L_+ . Finitely many isolated eigenvalues of the operators A and K^{-1} in \mathcal{H} are distributed between negative, zero and positive eigenvalues away of $\sigma_c(L_\pm)$. By the spectral theory of the self-adjoint operators, the Hilbert space \mathcal{H} can be equivalently decomposed into two orthogonal sums of subspaces which are invariant with respect to the operators K and A :

$$\mathcal{H} = \mathcal{H}_K^- \oplus \mathcal{H}_K^+ \oplus \mathcal{H}_K^{\sigma_e(K)}, \quad (2.4)$$

$$\mathcal{H} = \mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+ \oplus \mathcal{H}_A^{\sigma_e(A)}, \quad (2.5)$$

where notation $-(+)$ stands for the negative (positive) isolated eigenvalues, 0 for the isolated kernel, and σ_e for the essential spectrum that includes the absolute continuous part and embedded eigenvalues. It is clear that $\sigma_e(K)$ belongs to the interval $(0, \omega_-^{-1}]$ and $\sigma_e(A)$ belongs to the interval $[\omega_+, \infty)$. Since \mathcal{P} is a projection defined by eigenspaces of L_- while $K = \mathcal{P}L_-^{-1}\mathcal{P}$, it is clear that $\dim(\mathcal{H}_K^-) = n(L_-)$ and $\dim(\mathcal{H}_K^+) = p(L_-)$.

Proposition 2.1 *Let $\omega_+ > 0$. There exist $n_0 \geq 0$, $z_0 \geq 0$, and $z_1 \geq 0$, such that*

$$\dim(\mathcal{H}_A^-) = n(L_+) - n_0, \quad \dim(\mathcal{H}_A^0) = z(L_+) - z_0 + z_1, \quad \dim(\mathcal{H}_A^+) \leq p(L_+) + n_0 + z_0 - z_1. \quad (2.6)$$

Proof. Let $\ker(L_-) = \{v_1, v_2, \dots, v_n\} \in \mathcal{X}$ and define the matrix-valued function $A(\mu)$:

$$A_{ij}(\mu) = ((\mu - L_+)^{-1}v_i, v_j), \quad 1 \leq i, j \leq n$$

for all μ not in the spectrum of L_+ . When $z(L_+) = 0$, the first two equalities (2.6) follow by the abstract Lemma 3.4 in [CPV05], where n_0 is the number of nonnegative eigenvalues of $A(0)$, $z_0 = 0$, and z_1 is the number of zero eigenvalues of $A(0)$. When $z(L_+) \neq 0$, the same proof is extended for $A_0 = \lim_{\mu \rightarrow 0^-} A(\mu)$ where z_0 is the number of eigenvectors in the kernel of L_+ in \mathcal{X} which do not belong to the space \mathcal{H} (see proof of Theorem 2.9 in [CPV05]). The last inequality in (2.6) is obtained by extending the analysis of [CPV05] from $\mu = 0$ to $\mu = \omega_+ > 0$, where the upper bound is achieved if all eigenvalues of $A_+ = \lim_{\mu \rightarrow \omega_+} A(\mu)$ are negative. \blacksquare

Since A has finitely many negative eigenvalues and K has no kernel in \mathcal{H} , there exists a small number $\delta > 0$ in the gap $0 < \delta < |\sigma_{-1}|$, where σ_{-1} is the smallest (in absolute value) negative eigenvalue of $K^{-1}A$. The operator $A + \delta K$ is continuously invertible in \mathcal{H} and the generalized eigenvalue problem (2.3) is rewritten in the shifted form,

$$(A + \delta K)u = (\gamma + \delta)Ku, \quad u \in \mathcal{H}. \quad (2.7)$$

By the spectral theory, an alternative decomposition of the Hilbert space \mathcal{H} exists for $0 < \delta < |\sigma_{-1}|$:

$$\mathcal{H} = \mathcal{H}_{A+\delta K}^- \oplus \mathcal{H}_{A+\delta K}^+ \oplus \mathcal{H}_{A+\delta K}^{\sigma_e(A+\delta K)}, \quad (2.8)$$

where $\sigma_e(A + \delta K)$ belongs to the interval $[\omega_{A+\delta K}, \infty)$ and $\omega_{A+\delta K}$ is the minimum of $\sigma_c(A + \delta K)$. We will assume that $\omega_{A+\delta K} > 0$ for $\delta > 0$ even if $\omega_+ = 0$.

Proposition 2.2 *Let the number of negative (positive) eigenvalues of $A + \delta K$ bifurcating from the zero eigenvalues of A as $\delta > 0$ be denoted as n_- (n_+). When $\omega_+ > 0$, the splitting is complete so that*

$$\dim(\mathcal{H}_{A+\delta K}^\pm) = \dim(\mathcal{H}_A^\pm) + n_\pm, \quad \dim(\mathcal{H}_A^0) = n_- + n_+. \quad (2.9)$$

When $\omega_+ = 0$ and $\dim(\mathcal{H}_A^0) = 1$, the statement remains valid provided that the bifurcating eigenvalue of $A + \delta K$ is smaller than $\omega_{A+\delta K} > 0$.

Proof. For any $\delta \in (0, |\sigma_{-1}|)$, the operator $A + \delta K$ has no zero eigenvalues in \mathcal{H} . The equality (2.9) follows from the definition of n_- and n_+ . ■

The Pontryagin's Invariant Subspace theorem can be applied to the product of two bounded invertible self-adjoint operators $(A + \delta K)^{-1}$ and K in \mathcal{H} .

3 The proof of the Pontryagin's Invariant Subspace theorem

We shall develop an abstract theory of Pontryagin spaces with sign-indefinite metric, where the main result is the Pontryagin's Invariant Subspace theorem.

Definition 3.1 Let \mathcal{H} be a Hilbert space equipped with the inner product (\cdot, \cdot) and the sesquilinear form $[\cdot, \cdot]^1$. The Hilbert space \mathcal{H} is called the Pontryagin space (denoted as Π_κ) if it can be decomposed into the sum, which is orthogonal with respect to $[\cdot, \cdot]$,

$$\mathcal{H} \doteq \Pi_\kappa = \Pi_+ \oplus \Pi_-, \quad (3.1)$$

where Π_+ is a Hilbert space with the inner product $(\cdot, \cdot) = [\cdot, \cdot]$, Π_- is a Hilbert space with the inner product $(\cdot, \cdot) = -[\cdot, \cdot]$, and $\kappa = \dim(\Pi_-) < \infty$.

Remark 3.2 We shall write components of an element x in the Pontryagin space Π_κ as a vector $x = \{x_-, x_+\}$. The orthogonal sum (3.1) implies that any non-zero element $x \neq 0$ is represented by two terms,

$$\forall x \in \Pi_\kappa : \quad x = x_+ + x_-, \quad (3.2)$$

such that

$$[x_+, x_-] = 0, \quad [x_+, x_+] > 0, \quad [x_-, x_-] < 0, \quad (3.3)$$

and $\Pi_+ \cap \Pi_- = \emptyset$.

Definition 3.3 We say that Π is a non-positive subspace of Π_κ if $[x, x] \leq 0 \forall x \in \Pi$. We say that the non-positive subspace Π has the maximal dimension κ if any subspace of Π_κ of dimension higher than κ is not a non-positive subspace of Π_κ . Similarly, Π is a non-negative (neutral) subspace of Π_κ if $[x, x] \geq 0$ ($[x, x] = 0$) $\forall x \in \Pi$. The sign of $[x, x]$ on the element x of a subspace Π is called the Krein signature of the subspace Π .

Theorem 1 (Pontryagin) Let T be a self-adjoint bounded operator in Π_κ , such that $[T\cdot, \cdot] = [\cdot, T\cdot]$. There exists a T -invariant non-positive subspace of Π_κ of the maximal dimension κ .

¹We say that a complex-valued form $[u, v]$ on the product space $\mathcal{H} \times \mathcal{H}$ is a sesquilinear form if it is linear in u for each fixed v and linear with complex conjugate in v for each fixed u .

Remark 3.4 There are historically two completely different approaches to the proof of this theorem. A proof based on theory of analytic functions was given by Pontryagin [P44] while a proof based on angular operators was given by Krein [GK69] and later developed by students of M.G. Krein [AI86, IKL82]. Theorem 1 was rediscovered by Grillakis [G90] with the use of topology. We will describe a geometric proof of Theorem 1 based on the Fixed Point theorem. The proof uses the Cayley transformation of a self-adjoint operator in Π_κ to a unitary operator in Π_κ (Lemma 3.5) and the Krein's representation of the maximal non-positive subspace of Π_κ in terms of a graph of the contraction map (Lemma 3.7). While many statements of our analysis are available in the literature, details of the proofs are missing. Our presentation gives full details of the proof of Theorem 1 (see [GKP04] for a similar treatment in the case of compact operators).

Lemma 3.5 *Let T be a linear operator in Π_κ and $z \in \mathbb{C}$, $\text{Im}(z) > 0$ be a regular point of the operator T , such that $z \in \rho(T)$. Let U be the Cayley transform of T defined by $U = (T - \bar{z})(T - z)^{-1}$. The operators T and U have the same invariant subspaces in Π_κ .*

Proof. Let Π be a finite-dimensional invariant subspace of the operator T in Π_κ . It follows from $z \in \rho(T)$ that $(T - z)\Pi = \Pi$ then $(T - z)^{-1}\Pi = \Pi$ and $(T - \bar{z})(T - z)^{-1}\Pi \subseteq \Pi$, i.e $U\Pi \subseteq \Pi$. Conversely, let Π be an invariant subspace of the operator U . It follows from $U - I = (z - \bar{z})(T - z)^{-1}$ that $1 \in \rho(U)$ therefore $\Pi = (U - I)\Pi = (T - z)^{-1}\Pi$. From there, $\Pi \subseteq \text{dom}(T)$ and $(T - z)\Pi = \Pi$ so $T\Pi \subseteq \Pi$. ■

Corollary 3.6 *If T is a self-adjoint operator in Π_κ , then U is a unitary operator in Π_κ .*

Proof. We shall prove that $[Ug, Ug] = [g, g]$, where $g \in \text{dom}(U)$, by the explicit computation:

$$\begin{aligned} [Ug, Ug] &= [(T - \bar{z})f, (T - \bar{z})f] = [Tf, Tf] - z[f, Tf] - \bar{z}[Tf, f] + |z|^2[f, f], \\ [g, g] &= [(T - z)f, (T - z)f] = [Tf, Tf] - \bar{z}[f, Tf] - z[Tf, f] + |z|^2[f, f], \end{aligned}$$

where we have introduced $f \in \text{dom}(T)$ such that $f = (T - z)^{-1}g$. ■

Lemma 3.7 *A linear subspace $\Pi \subseteq \Pi_\kappa$ is a κ -dimensional non-positive subspace of Π_κ if and only if it is a graph of the contraction map $\mathcal{K} : \Pi_- \rightarrow \Pi_+$, such that $\Pi = \{x_-, \mathcal{K}x_-\}$ and $\|\mathcal{K}x_-\| \leq \|x_-\|$.*

Proof. Let $\Pi = \{x_-, x_+\}$ be a κ -dimensional non-positive subspace of Π_κ . We will show that there exist a contraction map $\mathcal{K} : \Pi_- \mapsto \Pi_+$ such that Π is a graph of \mathcal{K} . Indeed, the subspace Π is a graph of a linear operator \mathcal{K} if and only if it follows from $\{0, x_+\} \in \Pi$ that $x_+ = 0$. Since Π is non-positive with respect to $[\cdot, \cdot]$, then $[x, x] = \|x_+\|^2 - \|x_-\|^2 \leq 0$, where $\|\cdot\|$ is a norm in \mathcal{H} . As a result, $0 \leq \|x_+\| \leq \|x_-\|$ and if $x_- = 0$ then $x_+ = 0$. Moreover, for any $x_- \in \Pi_-$, it is true that $\|\mathcal{K}x_-\| \leq \|x_-\|$ such that \mathcal{K} is a

contraction map. Conversely, let \mathcal{K} be a contraction map $\mathcal{K} : \Pi_- \mapsto \Pi_+$. The graph of \mathcal{K} belongs to the non-positive subspace of Π_κ as

$$[x, x] = \|x_+\|^2 - \|x_-\|^2 = \|\mathcal{K}x_-\|^2 - \|x_-\|^2 \leq 0.$$

Let $\Pi = \{x_-, \mathcal{K}x_-\}$. Since $\dim(\Pi_-) = \kappa$, then $\dim(\Pi) = \kappa$.² ■

Proof of Theorem 1. Let $z \in \mathbb{C}$ and $\text{Im}(z) > 0$. Then, z is a regular point of the self-adjoint operator T in Π_κ . Let $U = (T - \bar{z})(T - z)^{-1}$ be the Cayley transform of T . By Corollary 3.6, U is a unitary operator in Π_κ . By Lemma 3.5, T and U have the same invariant subspaces in Π_κ . Therefore, the existence of the maximal non-positive invariant subspace for the self-adjoint operator T can be proved from the existence of such a subspace for the unitary operator U . Let $x = \{x_-, x_+\}$ and

$$U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$$

be the matrix representation of the operator U with respect to the decomposition (3.1). Let Π denote a κ -dimensional non-positive subspace in Π_κ . Since U has an empty kernel in Π_κ and U is unitary in Π_κ such that $[Ux_-, Ux_-] = [x_-, x_-] \leq 0$, then $\tilde{\Pi} = U\Pi$ is also a κ -dimensional non-positive subspace of Π_κ . By Lemma 3.7, there exist two contraction mappings \mathcal{K} and $\tilde{\mathcal{K}}$ for subspaces Π and $\tilde{\Pi}$, respectively. Therefore, the assignment $\tilde{\Pi} = U\Pi$ is equivalent to the system,

$$\begin{pmatrix} \tilde{x}_- \\ \tilde{\mathcal{K}}\tilde{x}_- \end{pmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{pmatrix} x_- \\ \mathcal{K}x_- \end{pmatrix} = \begin{pmatrix} (U_{11} + U_{12}\mathcal{K})x_- \\ (U_{21} + U_{22}\mathcal{K})x_- \end{pmatrix},$$

such that

$$U_{21} + U_{22}\mathcal{K} = \tilde{\mathcal{K}}(U_{11} + U_{12}\mathcal{K}).$$

We shall prove that the operator $(U_{11} + U_{12}\mathcal{K})$ is invertible. By contradiction, we assume that there exists $x_- \neq 0$ such that $\tilde{x}_- = (U_{11} + U_{12}\mathcal{K})x_- = 0$. Since $\tilde{x}_- = 0$ implies that $\tilde{x}_+ = \tilde{\mathcal{K}}\tilde{x}_- = 0$, we obtain that $[x_-, \mathcal{K}x_-]^T$ is an eigenvector in the kernel of U . However, U has an empty kernel in Π_κ such that $x_- = 0$. Let $F(\mathcal{K})$ be an operator-valued function in the form,

$$F(\mathcal{K}) = (U_{21} + U_{22}\mathcal{K})(U_{11} + U_{12}\mathcal{K})^{-1},$$

such that $\tilde{\mathcal{K}} = F(\mathcal{K})$. By Lemma 3.7, the operator $F(\mathcal{K})$ maps the operator unit ball $\|\mathcal{K}\| \leq 1$ to itself. Since U is a continuous operator and U_{12} is a finite-dimensional operator, then U_{12} is a compact operator. Hence the operator ball $\|\mathcal{K}\| \leq 1$ is a weakly compact set and the function $F(\mathcal{K})$ is continuous with respect to weak topology. By the Schauder's Fixed-Point Principle, there exists a fixed point \mathcal{K}_0 such that

²Extending arguments of Lemma 3.7, one can prove that the subspace Π is strictly negative with respect to $[\cdot, \cdot]$ if and only if it is a graph of the strictly contraction map $\mathcal{K} : \Pi_- \mapsto \Pi_+$, such that $\Pi = \{x_-, \mathcal{K}x_-\}$ and $\|\mathcal{K}x_-\| < \|x_-\|$.

$F(\mathcal{K}_0) = \mathcal{K}_0$ and $\|\mathcal{K}_0\| \leq 1$. By Lemma 3.7, the graph of \mathcal{K}_0 defines the κ -dimensional non-positive subspace Π , which is invariant with respect to U .

It remains to prove that the κ -dimensional non-positive subspace Π has the maximal dimension that a non-positive subspace of Π_κ can have. By contradiction, we assume that there exists a $(\kappa + 1)$ -dimensional non-positive subspace $\tilde{\Pi}$. Let $\{e_1, e_2, \dots, e_\kappa\}$ be a basis in Π_- in the canonical decomposition (3.2). We fix two elements $y_1, y_2 \in \tilde{\Pi}$ with the same projections to $\{e_1, e_2, \dots, e_\kappa\}$, such that

$$\begin{aligned} y_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_\kappa e_\kappa + y_{1p}, \\ y_2 &= \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_\kappa e_\kappa + y_{2p}, \end{aligned}$$

where $y_{1p}, y_{2p} \in \Pi_+$. It is clear that $y_1 - y_2 = y_{1p} - y_{2p} \in \Pi_+$ such that $[y_{1p} - y_{2p}, y_{1p} - y_{2p}] > 0$. On the other hand, $y_1 - y_2 \in \tilde{\Pi}$, such that $[y_1 - y_2, y_1 - y_2] \leq 0$. Hence $y_{1p} = y_{2p}$ and then $y_1 = y_2$. We proved that any vector in $\tilde{\Pi}$ is uniquely determined by the κ -dimensional basis $\{e_1, e_2, \dots, e_\kappa\}$ and the non-positive subspace $\tilde{\Pi}$ is hence κ -dimensional. ■

4 Bounds on eigenvalues of the generalized eigenvalue problem

We shall apply Theorem 1 to the product of two bounded invertible self-adjoint operators $B = (A + \delta K)^{-1}$ and K , where $\delta \in (0, |\sigma_{-1}|)$ and σ_{-1} is the smallest negative eigenvalue of $K^{-1}A$. Properties of self-adjoint operators A and K^{-1} in \mathcal{H} follow from properties of self-adjoint operators L_\pm in \mathcal{X} , which are summarized in the main assumptions P1–P2. With a slight abuse of notations, we shall denote eigenvalues of the operator $T = BK$ by λ , which is expressed in terms of the eigenvalue γ of the shifted generalized eigenvalue problem (2.7) by $\lambda = (\gamma + \delta)^{-1}$. We note that λ here does not correspond to λ used in the linearized Hamiltonian problem (2.1).

Lemma 4.1 *Let \mathcal{H} be a Hilbert space with the inner product (\cdot, \cdot) and $B, K : \mathcal{H} \rightarrow \mathcal{H}$ be bounded invertible self-adjoint operators in \mathcal{H} . Define the sesquilinear form $[\cdot, \cdot] = (K\cdot, \cdot)$ and extend \mathcal{H} to the Pontryagin space Π_κ , where κ is the finite number of negative eigenvalues of K counted with their multiplicities. The operator $T = BK$ is self-adjoint in Π_κ and there exists a κ -dimensional non-positive subspace of Π_κ which is invariant with respect to T .*

Proof. It follows from the orthogonal sum decomposition (2.4) that the quadratic form $(K\cdot, \cdot)$ is strictly negative on the κ -dimensional subspace \mathcal{H}_K^- and strictly positive on the infinite-dimensional subspace $\mathcal{H}_K^+ \oplus \mathcal{H}_K^{\sigma_e(K)}$. By continuity and Gram–Schmidt orthogonalization, the Hilbert space \mathcal{H} is extended to the Pontryagin space Π_κ with respect to the sesquilinear form $[\cdot, \cdot] = (K\cdot, \cdot)$. The bounded operator $T = BK$ is self-adjoint in Π_κ , since B and K are self-adjoint in \mathcal{H} and

$$[T\cdot, \cdot] = (KBK\cdot, \cdot) = (K\cdot, BK\cdot) = [\cdot, T\cdot].$$

Existence of the κ -dimensional non-positive T -invariant subspace of Π_κ follows from Theorem 1. \blacksquare

Remark 4.2 The decomposition (3.1) of the Pontryagin space Π_κ is canonical in the sense that $\Pi_+ \cap \Pi_- = \emptyset$. We will now consider various sign-definite subspaces of Π_κ which are invariant with respect to the self-adjoint operator $T = BK$ in Π_κ . In general, these invariant sign-definite subspaces do not provide a canonical decomposition of Π_κ . Let us denote the invariant subspace of T associated with complex eigenvalues in the upper (lower) half-plane as \mathcal{H}_{c+} (\mathcal{H}_{c-}) and the non-positive (non-negative) invariant subspace of T associated with real eigenvalues as $\mathcal{H}_n(\mathcal{H}_p)$. The invariant subspace \mathcal{H}_n , prescribed by Lemma 4.1, may include both isolated and embedded eigenvalues of T in Π_κ . We will show that this subspace does not include the residual and absolutely continuous parts of the spectrum of T in Π_κ .

4.1 Residual and absolutely continuous spectra of T in Π_κ

Definition 4.3 We say that λ is a point of the residual spectrum of T in Π_κ if $\text{Ker}(T - \lambda I) = \emptyset$ but $\overline{\text{Range}(T - \lambda I)} \neq \Pi_\kappa$ and λ is a point of the continuous spectrum of T in Π_κ if $\text{Ker}(T - \lambda I) = \emptyset$ but $\text{Range}(T - \lambda I) \neq \overline{\text{Range}(T - \lambda I)} = \Pi_\kappa$.

Lemma 4.4 No residual part of the spectrum of T in Π_κ exists.

Proof. By a contradiction, assume that λ belongs to the residual part of the spectrum of T in Π_κ such that $\text{Ker}(T - \lambda I) = \emptyset$ but $\text{Range}(T - \lambda I)$ is not dense in Π_κ . Let $g \in \Pi_\kappa$ be orthogonal to $\text{Range}(T - \lambda I)$, such that

$$\forall f \in \Pi_\kappa : \quad 0 = [(T - \lambda I)f, g] = [f, (T - \bar{\lambda}I)g].$$

Therefore, $(T - \bar{\lambda}I)g = 0$, that is $\bar{\lambda}$ is an eigenvalue of T . By symmetry of eigenvalues, λ is also an eigenvalue of T and hence it can not be in the residual part of the spectrum of T . \blacksquare

Remark 4.5 It is assumed in [G90] that the residual part of spectrum is empty and that the kernels of operators A and K are empty. The first assumption is now proved in Lemma 4.4 and the second assumption is removed with the use of the shifted generalized eigenvalue problem (2.7).

Lemma 4.6 The absolutely continuous part of the spectrum of T in Π_κ is real.

Proof. Let P^+ and P^- be orthogonal projectors to Π^+ and Π^- respectively, such that $I = P^+ + P^-$. Since Π^\pm are defined by $(\cdot, K\cdot)$, the self-adjoint operator K admits the polar decomposition $K = J|K|$, where $J = P^+ - P^-$ and $|K|$ is a positive operator. The operator $T = BK$ is similar to the operator

$$|K|^{1/2}BJ|K|^{1/2} = |K|^{1/2}BJ|K|^{1/2}(J + 2P^-) = |K|^{1/2}B|K|^{1/2} + 2|K|^{1/2}BJ|K|^{1/2}P^-.$$

Since P^- is a projection to a finite-dimension subspace, the operator $|K|^{1/2}BJ|K|^{1/2}$ is the finite-dimensional perturbation of the self-adjoint operator $|K|^{1/2}B|K|^{1/2}$. By perturbation theory [K76], the absolutely continuous part of the spectrum of the self-adjoint operator $|K|^{1/2}B|K|^{1/2}$ is the same as that of $|K|^{1/2}BJ|K|^{1/2}$. By similarity transformation, it is the same as that of T . ■

Lemma 4.7 (Cauchy-Schwartz) *Let Π be either non-positive or non-negative subspace of Π_κ . Then,*

$$\forall f, g \in \Pi : \quad |[f, g]|^2 \leq [f, f][g, g]. \quad (4.1)$$

Proof. The proof resembles that of the standard Cauchy–Schwartz inequality. Let Π be a non-positive subspace of Π_κ . Then, for any $f, g \in \Pi$ and any $\alpha, \beta \in \mathbb{C}$, we have $[\alpha f + \beta g, \alpha f + \beta g] \leq 0$ and

$$[\alpha f + \beta g, \alpha f + \beta g] = [f, f]|\alpha|^2 + [f, g]\alpha\bar{\beta} + [g, f]\bar{\alpha}\beta + [g, g]|\beta|^2. \quad (4.2)$$

If $[f, g] = 0$, then (4.1) is satisfied as $[f, f] \leq 0$ and $[g, g] \leq 0$. If $[f, g] \neq 0$, let $\alpha \in \mathbb{R}$ and

$$\beta = \frac{[f, g]}{|[f, g]|},$$

such that

$$[f, f]\alpha^2 + 2\alpha|[f, g]| + [g, g] \leq 0.$$

The last condition is satisfied if the discriminant of the quadratic equation is non-positive such that $4|[f, g]|^2 - 4[f, f][g, g] \leq 0$, that is (4.1). Let Π be a non-negative subspace of Π_κ . Then, for any $f, g \in \Pi$ and any $\alpha, \beta \in \mathbb{C}$, we have $[\alpha f + \beta g, \alpha f + \beta g] \geq 0$ and the same arguments result in the same inequality (4.1). ■

Corollary 4.8 *Let Π be either non-positive or non-negative subspace of Π_κ . Let $f \in \Pi$ such that $[f, f] = 0$. Then $[f, g] = 0 \forall g \in \Pi$.*

Proof. The proof follows from (4.1) since $0 \leq |[f, g]|^2 \leq 0$. ■

Lemma 4.9 *Let Π be an invariant subspace of Π_κ with respect to operator T and Π^\perp be the orthogonal compliment of Π in Π_κ with respect to $[\cdot, \cdot]$. Then, Π^\perp is also invariant with respect to T .*

Proof. For all $f \in \text{Dom}(T) \cap \Pi$, we have $Tf \in \Pi$. Let $g \in \text{Dom}(T) \cap \Pi^\perp$. Then $[g, Tf] = [Tg, f] = 0$. ■

Theorem 2 *Let Π_c be a subspace related to the absolute continuous spectrum of T in Π_κ . Then $[f, f] > 0 \forall f \in \Pi_c$.*

Proof. By contradiction, assume that there exists $f_0 \in \Pi_c$ such that $[f_0, f_0] < 0$. Since Π_c is a subspace for the absolutely continuous spectrum, there exists a continuous family of functions $f_\alpha \in \Pi_c$ such that $[f_\alpha, f_\alpha] < 0$ and hence $f_\alpha \in \Pi_-$ in the decomposition (3.1). However, this contradicts to the fact that $\dim(\Pi_-) = \kappa < \infty$. Therefore, Π_c is a non-negative subspace of Π_κ . Assume that there exists an element $f_0 \in \Pi_c$ such that $[f_0, f_0] = 0$. By Corollary 4.8, $f_0 \in \Pi_c^\perp$. By Lemma 4.9, Π_c^\perp is an invariant subspace of Π_κ . The intersection of invariant subspaces is invariant, such that $\Pi_c \cap \Pi_c^\perp$ is a neutral invariant subspace of Π_κ . By Lemma 4.1, $\dim(\Pi_c \cap \Pi_c^\perp) \leq \kappa$. Therefore, f_0 is an element of a finite-dimensional invariant subspace of Π_κ , which is a contradiction to the fact that $f_0 \in \Pi_c$. ■

4.2 Isolated and embedded eigenvalues of T in Π_κ

Definition 4.10 We say that λ is an eigenvalue of T in Π_κ if $\text{Ker}(T - \lambda I) \neq \emptyset$. The eigenvalue of T is said to be semi-simple if it is not simple and the algebraic and geometric multiplicities coincide. Otherwise, the non-simple eigenvalue is said to be multiple. Let λ_0 be an eigenvalue of T with algebraic multiplicity n and geometric multiplicity one. The canonical basis for the corresponding n -dimensional eigenspace of T is defined by the Jordan chain of eigenvectors,

$$f_j \in \Pi_\kappa : \quad T f_j = \lambda_0 f_j + f_{j-1}, \quad j = 1, \dots, n, \quad (4.3)$$

where $f_0 = 0$.

Lemma 4.11 Let \mathcal{H}_λ and \mathcal{H}_μ be eigenspaces of the eigenvalues λ and μ of the operator T in Π_κ and $\lambda \neq \bar{\mu}$. Then \mathcal{H}_λ is orthogonal to \mathcal{H}_μ with respect to $[\cdot, \cdot]$.

Proof. Let n and m be dimensions of \mathcal{H}_λ and \mathcal{H}_μ , respectively, such that $n \geq 1$ and $m \geq 1$. By Definition 4.10, it is clear that

$$f \in \mathcal{H}_\lambda \iff (T - \lambda I)^n f = 0, \quad (4.4)$$

$$g \in \mathcal{H}_\mu \iff (T - \mu I)^m g = 0. \quad (4.5)$$

We should prove that $[f, g] = 0$ by induction for $n + m \geq 2$. If $n + m = 2$ ($n = m = 1$), then it follows from the system (4.4)–(4.5) that

$$(\lambda - \bar{\mu})[f, g] = 0, \quad f \in \mathcal{H}_\lambda, \quad g \in \mathcal{H}_\mu,$$

such that $[f, g] = 0$ for $\lambda \neq \bar{\mu}$. Let us assume that subspaces \mathcal{H}_λ and \mathcal{H}_μ are orthogonal for $2 \leq n + m \leq k$ and prove that extended subspaces $\tilde{\mathcal{H}}_\lambda$ and $\tilde{\mathcal{H}}_\mu$ remain orthogonal for $\tilde{n} = n + 1$, $\tilde{m} = m$ and $\tilde{n} = n$, $\tilde{m} = m + 1$. In either case, we define $\tilde{f} = (T - \lambda I)f$ and $\tilde{g} = (T - \mu I)g$, such that

$$f \in \tilde{\mathcal{H}}_\lambda \iff (T - \lambda I)^{\tilde{n}} \tilde{f} = (T - \lambda I)^n \tilde{f} = 0,$$

$$g \in \tilde{\mathcal{H}}_\mu \iff (T - \mu I)^{\tilde{m}} \tilde{g} = (T - \mu I)^m \tilde{g} = 0.$$

By the inductive assumption, we have $[\tilde{f}, g] = 0$ and $[f, \tilde{g}] = 0$ in either case, such that

$$[(T - \lambda I)f, g] = 0 \quad [f, (T - \mu I)g] = 0. \quad (4.6)$$

By using the system (4.4)–(4.5) and the relations (4.6), we obtain that

$$(\lambda - \bar{\mu})[f, g] = 0, \quad f \in \tilde{\mathcal{H}}_\lambda, \quad g \in \tilde{\mathcal{H}}_\mu,$$

from which the statement follows by the induction method. \blacksquare

Lemma 4.12 *Let \mathcal{H}_{λ_0} be eigenspace of a multiple real isolated eigenvalue λ_0 of T in Π_κ and $\{f_1, f_2, \dots, f_n\}$ be the Jordan chain of eigenvectors. Let $\mathcal{H}_0 = \text{span}\{f_1, f_2, \dots, f_k\} \subset \mathcal{H}_{\lambda_0}$, where $k = \text{round}(n/2)$, and $\tilde{\mathcal{H}}_0 = \text{span}\{f_1, f_2, \dots, f_k, f_{k+1}\} \subset \mathcal{H}_{\lambda_0}$.*

- *If n is even ($n = 2k$), the neutral subspace \mathcal{H}_0 is the maximal sign-definite subspace of \mathcal{H}_{λ_0} .*
- *If n is odd ($n = 2k+1$), the subspace $\tilde{\mathcal{H}}_0$ is the maximal non-negative subspace of Π_κ if $[f_1, f_n] > 0$ and the maximal non-positive subspace of Π_κ if $[f_1, f_n] < 0$, while the neutral subspace \mathcal{H}_0 is the maximal non-positive subspace of Π_κ if $[f_1, f_n] > 0$ and the maximal non-negative subspace of Π_κ if $[f_1, f_n] < 0$.*

Proof. Without loss of generality we will consider the case $\lambda_0 = 0$ (if $\lambda_0 \neq 0$ the same argument is applied to the shifted self-adjoint operator $\tilde{T} = T - \lambda_0 I$). We will show that $[f, f] = 0 \forall f \in \mathcal{H}_0$. By a decomposition over the basis in \mathcal{H}_0 , we obtain

$$[f, f] = \sum_{i=1}^k \sum_{j=1}^k \alpha_i \bar{\alpha}_j [f_i, f_j]. \quad (4.7)$$

We use that

$$[f_i, f_j] = [Tf_{i+1}, Tf_{j+1}] = \dots = [T^k f_{i+k}, T^k f_{j+k}] = [T^{2k} f_{i+k}, f_{j+k}],$$

for any $1 \leq i, j \leq k$. In case of even $n = 2k$, we have $[f_i, f_j] = [T^n f_{i+k}, f_{j+k}] = 0$ for all $1 \leq i, j \leq k$. In case of odd $n = 2k+1$, we have $[f_i, f_j] = [T^{n+1} f_{i+k+1}, f_{j+k+1}] = 0$ for all $1 \leq i, j \leq k$. Therefore, \mathcal{H}_0 is a neutral subspace of \mathcal{H}_{λ_0} . To show that it is actually the maximal neutral subspace of \mathcal{H}_{λ_0} , let $\mathcal{H}'_0 = \text{span}\{f_1, f_2, \dots, f_k, f_{k_0}\}$, where $k+1 \leq k_0 \leq n$. Since f_{n+1} does not exist in the Jordan chain (4.3) (otherwise, the algebraic multiplicity is $n+1$) and λ_0 is an isolated eigenvalue, then $[f_1, f_n] \neq 0$ by the Fredholm theory. It follows from the Jordan chain (4.3) that

$$[f_1, f_n] = [T^{m-1} f_m, f_n] = [f_m, T^{m-1} f_n] = [f_m, f_{n-m+1}] \neq 0. \quad (4.8)$$

When $n = 2k$, we have $1 \leq n - k_0 + 1 \leq k$, such that $[f_{k_0}, f_{n-k_0+1}] \neq 0$ and the subspace \mathcal{H}'_0 is sign-indefinite in the decomposition (4.7). When $n = 2k+1$, we have $1 \leq n - k_0 + 1 \leq k$ for $k_0 \geq k+2$ and

$n - k_0 + 1 = k + 1$ for $k_0 = k + 1$. In either case, $[f_{k_0}, f_{n-k_0+1}] \neq 0$ and the subspace \mathcal{H}'_0 is sign-indefinite in the decomposition (4.7) unless $k_0 = k + 1$. In the latter case, we have $[f_{k+1}, f_{k+1}] = [f_1, f_n] \neq 0$ and $[f_j, f_{k+1}] = [T^{2k} f_{j+k}, f_n] = 0$ for $1 \leq j \leq k$. As a result, the subspace $\tilde{\mathcal{H}}_0 \equiv \mathcal{H}'_0$ is non-negative for $[f_1, f_n] > 0$ and non-positive for $[f_1, f_n] < 0$. ■

Corollary 4.13 *Let \mathcal{H}_{λ_0} be eigenspace of a semi-simple real isolated eigenvalue λ_0 of T in Π_κ . Then, the eigenspace \mathcal{H}_{λ_0} is either strictly positive or strictly negative subspace of Π_κ with respect to $[\cdot, \cdot]$.*

Proof. The proof follows by contradiction. Let \tilde{f} be a particular linear combination of eigenvectors in \mathcal{H}_{λ_0} , such that $[\tilde{f}, \tilde{f}] = 0$. By the Fredholm theory, there exists a solution of the Jordan chain (4.3), where $f_1 \equiv \tilde{f}$. Then, the eigenvalue λ_0 could not be semi-simple. ■

Remark 4.14 If λ_0 is a real (multiple or semi-simple) embedded eigenvalue of T , the Jordan chain can be truncated at f_n even if $[f_1, f_n] = 0^3$. In the latter case, the neutral subspaces \mathcal{H}_0 for $n = 2k$ and $\tilde{\mathcal{H}}_0$ for $n = 2k + 1$ in Lemma 4.12 do not have to be maximal non-positive or non-negative subspaces, while the subspace \mathcal{H}_{λ_0} in Corollary 4.13 could be sign-indefinite. The construction of a maximal non-positive subspace for (multiple or semi-simple) embedded eigenvalues depend on the computations of the projection matrix $[f_i, f_j]$ in the eigenspace of eigenvectors $\mathcal{H}_\lambda = \text{span}\{f_1, \dots, f_n\}$. We shall simplify this unnecessary complication by an assumption that *all embedded eigenvalues are simple*, such that the corresponding one-dimensional eigenspace \mathcal{H}_{λ_0} is either positive or negative or neutral with respect to $[\cdot, \cdot]$.

Remark 4.15 By Lemma 4.12 and Corollary 4.13, if λ_0 is a real (multiple or semi-simple) isolated eigenvalue, then the sum of dimensions of the maximal non-positive and non-negative subspaces of \mathcal{H}_{λ_0} equals the dimension of \mathcal{H}_{λ_0} (although the intersection of the two subspaces can be non-empty). By Remark 4.14, the dimension of \mathcal{H}_{λ_0} for a real embedded eigenvalue can however be smaller than the sum of dimensions of the maximal non-positive and non-negative subspaces. Therefore, the presence of embedded eigenvalues introduces a complication in the count of eigenvalues of the generalized eigenvalue problem (2.3). This complication was neglected in the implicit count of embedded eigenvalues in [KKS04].

Lemma 4.16 *Let $\lambda_0 \in \mathbb{C}$, $\text{Im}(\lambda_0) > 0$ be an eigenvalue of T in Π_κ , \mathcal{H}_{λ_0} be the corresponding eigenspace, and $\tilde{\mathcal{H}}_{\lambda_0} = \{\mathcal{H}_{\lambda_0}, \mathcal{H}_{\bar{\lambda}_0}\} \subset \Pi_\kappa$. Then, the neutral subspace \mathcal{H}_{λ_0} is the maximal sign-definite subspace of $\tilde{\mathcal{H}}_{\lambda_0}$, such that $[f, f] = 0 \forall f \in \mathcal{H}_{\lambda_0}$.*

Proof. By Lemma 4.11 with $\lambda = \mu = \lambda_0$, the eigenspace \mathcal{H}_{λ_0} is orthogonal to itself with respect to $[\cdot, \cdot]$, such that \mathcal{H}_{λ_0} is a neutral subspace of $\tilde{\mathcal{H}}_{\lambda_0}$. It remains to prove that \mathcal{H}_{λ_0} is the maximal sign-definite subspace in $\tilde{\mathcal{H}}_\lambda$. Let $\mathcal{H}_{\lambda_0} = \text{span}\{f_1, f_2, \dots, f_n\}$, where $\{f_1, f_2, \dots, f_n\}$ is the Jordan chain of

³The Fredholm theory gives a necessary but not a sufficient condition for existence of the solution f_{n+1} in the Jordan chain (4.3) if the eigenvalue λ_0 is embedded into the continuous spectrum.

eigenvectors (4.3). Consider a subspace $\tilde{H}_0 = \text{span}\{f_1, f_2, \dots, f_n, \bar{f}_j\}$, where $1 \leq j \leq n$ and construct a linear combination of f_{n+1-j} and \bar{f}_j :

$$[f_{n+1-j} + \alpha \bar{f}_j, f_{n+1-j} + \alpha \bar{f}_j] = 2\text{Re}(\alpha [\bar{f}_j, f_{n+1-j}]), \quad \alpha \in \mathbb{C}.$$

By the Fredholm theory, $[f_n, \bar{f}_1] \neq 0$ and by virtue of (4.8), $[\bar{f}_j, f_{n+1-j}] \neq 0$. As a result, the linear combination $f_{n+1-j} + \alpha \bar{f}_j$ is sign-indefinite with respect to $[\cdot, \cdot]$. ■

Lemma 4.17 *Let \mathcal{H}_0 be eigenspace of a multiple zero eigenvalue of $K^{-1}A$ in \mathcal{H} and $\{f_1, \dots, f_n\}$ be the Jordan chain of eigenvectors, such that $f_1 \in \text{Ker}(A)$. Let $0 < \delta < |\sigma_{-1}|$, where σ_{-1} is the smallest negative eigenvalue of $K^{-1}A$. If $\omega_+ > 0$ then $[f_1, f_n] = (Kf_1, f_n) \neq 0$, and*

- *If n is odd, the subspace \mathcal{H}_0 corresponds to a positive eigenvalue of the operator $(A + \delta K)$ if $[f_1, f_n] > 0$ and to a negative eigenvalue if $[f_1, f_n] < 0$.*
- *If n is even, the subspace \mathcal{H}_0 corresponds to a positive eigenvalue of the operator $(A + \delta K)$ if $[f_1, f_n] < 0$ and to a negative eigenvalue if $[f_1, f_n] > 0$.*

Proof. Let $\mu(\delta)$ be an eigenvalue of the operator $A + \delta K$ related to the subspace \mathcal{H}_0 . By standard perturbation theory for isolated eigenvalues [K76], each eigenvalue $\mu_j(\delta)$ is a continuous function of δ and

$$\lim_{\delta \rightarrow 0^+} \frac{\mu(\delta)}{\delta^n} = (-1)^{n+1} \frac{(Kf_1, f_n)}{(f_1, f_1)}. \quad (4.9)$$

If $\omega_+ > 0$, the zero eigenvalue of A is isolated from the essential spectrum of $K^{-1}A$, such that $[f_1, f_n] = (Kf_1, f_n) \neq 0$ by the Fredholm theory. The statement of the lemma follows from the limiting relation (4.9). Since no eigenvalues of $K^{-1}A$ exists in $(-\sigma_{-1}, 0)$, the eigenvalue $\mu(\delta)$ remains sign-definite for $0 < \delta < |\sigma_{-1}|$. ■

Remark 4.18 The statement holds for the case $\omega_+ = 0$ provided that $n = 1$, $[f_1, f_1] \neq 0$, and $\mu(\delta) < \omega_{A+\delta K}$, where $\omega_{A+\delta K} > 0$ is defined below the decomposition (2.8).

Remark 4.19 We are concerned here in the bounds on the numbers of eigenvalues in the generalized eigenvalue problem (2.3) in terms of the numbers of isolated eigenvalues of self-adjoint operators A and K^{-1} . Let N_p^- (N_n^-) be the number of negative eigenvalues of the bounded or unbounded operator $K^{-1}A$ with the account of their multiplicities whose eigenvectors are associated to the maximal non-negative (non-positive) subspace of Π_κ with respect to $[\cdot, \cdot] = (K\cdot, \cdot)$. Similarly, let N_p^0 (N_n^0) be the number of zero eigenvalues of $K^{-1}A$ with the account of their multiplicities and N_p^+ (N_n^+) be the number of positive eigenvalues of $K^{-1}A$ with the account of their multiplicities, such that the corresponding eigenvectors are associated to the maximal non-negative (non-positive) subspace of Π_κ . In the case of real isolated

eigenvalues, the sum of dimensions of the maximal non-positive and non-negative subspaces equals the dimension of the subspace \mathcal{H}_{λ_0} by Remark 4.15. The splitting of the dimension of \mathcal{H}_{λ_0} between $N_p(\lambda_0)$ and $N_n(\lambda_0)$ is obvious for each semi-simple isolated eigenvalue λ_0 in Corollary 4.13. In the case of a multiple real isolated eigenvalue λ_0 of algebraic multiplicity n , the same splitting is prescribed in Lemma 4.12:

- (i) If $n = 2k$, $k \in \mathbb{N}$, then $N_p(\lambda_0) = N_n(\lambda_0) = k$.
- (ii) If $n = 2k + 1$, $k \in \mathbb{N}$ and $[f_1, f_n] > 0$, then $N_p(\lambda_0) = k + 1$ and $N_n(\lambda_0) = k$.
- (iii) If $n = 2k + 1$, $k \in \mathbb{N}$ and $[f_1, f_n] < 0$, then $N_p(\lambda_0) = k$ and $N_n(\lambda_0) = k + 1$.

In the case of a simple real embedded eigenvalue λ_0 , the numbers $N_p(\lambda_0)$ and $N_n(\lambda_0)$ for a one-dimensional subspace \mathcal{H}_{λ_0} are prescribed in Remark 4.14 as follows:

- (i) If $[f_1, f_1] > 0$, then $N_p(\lambda_0) = 1$, $N_n(\lambda_0) = 0$.
- (ii) If $[f_1, f_1] < 0$, then $N_p(\lambda_0) = 0$, $N_n(\lambda_0) = 1$.
- (iii) If $[f_1, f_1] = 0$, then $N_p(\lambda_0) = N_n(\lambda_0) = 1$.

In the case (iii), the sum $N_p(\lambda_0) + N_n(\lambda_0)$ exceeds the dimension of \mathcal{H}_{λ_0} . Let N_{c+} (N_{c-}) be the number of complex eigenvalues in the upper half plane $\gamma \in \mathbb{C}$, $\text{Im}(\gamma) > 0$ ($\text{Im}(\gamma) < 0$). The maximal sign-definite subspace of Π_κ associated to complex eigenvalues is prescribed by Lemma 4.16.

Theorem 3 *Let assumptions P1–P2 be satisfied. Eigenvalues of the generalized eigenvalue problem (2.3) satisfy the pair of equalities:*

$$N_p^- + N_n^0 + N_n^+ + N_{c+} = \dim(\mathcal{H}_{A+\delta K}^-) \quad (4.10)$$

$$N_n^- + N_n^0 + N_n^+ + N_{c+} = \dim(\mathcal{H}_K^-) \quad (4.11)$$

Proof. We use the shifted eigenvalue problem (2.7) with sufficiently small $\delta > 0$ and consider the bounded operator $T = (A + \delta K)^{-1}K$. By Lemma 4.1, the operator T is self-adjoint with respect to $[\cdot, \cdot] = (K\cdot, \cdot)$ and it has a non-positive invariant subspace of dimension $\kappa = \dim(\mathcal{H}_K^-)$. Counting all eigenvalues of the shifted eigenvalue problem (2.7) in Remark 4.19, we establish the equality (4.11). The other equality (4.10) follows from a count for the bounded operator $\tilde{T} = K(A + \delta K)^{-1}$ which is self-adjoint with respect to $[\cdot, \cdot] = ((A + \delta K)^{-1}\cdot, \cdot)$. The self-adjoint operator $(A + \delta K)^{-1}$ defines the indefinite metric in the Pontryagin space $\tilde{\Pi}_{\tilde{\kappa}}$, where $\tilde{\kappa} = \dim(\mathcal{H}_{A+\delta K}^-)$. For any semi-simple eigenvalue γ_0 of the shifted eigenvalue problem (2.7), we have

$$\forall f, g \in \mathcal{H}_{\gamma_0}, \quad ((A + \delta K)f, g) = (\gamma_0 + \delta)(Kf, g).$$

If $\gamma_0 \geq 0$ or $\text{Im}(\gamma_0) \neq 0$, the maximal non-positive eigenspace of \tilde{T} in $\tilde{\Pi}_\kappa$ associated with γ_0 coincides with the maximal non-positive eigenspace of T in Π_κ . If $\gamma_0 < 0$, the maximal non-positive eigenspace of \tilde{T} in $\tilde{\Pi}_\kappa$ coincides with the maximal non-negative eigenspace of T in Π_κ . The same statement can be proved for the case of multiple isolated eigenvalues γ_0 when the eigenspace is defined by the Jordan block of eigenvectors,

$$Af_j = \gamma_0 K f_j + f_{j-1}, \quad j = 1, \dots, n,$$

where $f_0 = 0$. The dimension of the maximal non-positive eigenspace of \tilde{T} in $\tilde{\Pi}_\kappa$ is then $N_p^- + N_n^0 + N_n^+ + N_{c^+}$. ■

Corollary 4.20 *Let $N_{\text{neg}} = \dim(\mathcal{H}_{A+\delta K}^-) + \dim(\mathcal{H}_K^-)$ be the total negative index of the shifted generalized eigenvalue problem (2.7). Let $N_{\text{unst}} = N_p^- + N_n^- + 2N_{c^+}$ be the total number of unstable eigenvalues that include $N^- = N_p^- + N_n^-$ negative eigenvalues $\gamma < 0$ and $N_c = N_{c^+} + N_{c^-}$ complex eigenvalues with $\text{Im}(\gamma) \neq 0$. Under the assumptions of Theorem 3, it is true that*

$$\Delta N = N_{\text{neg}} - N_{\text{unst}} = 2N_n^+ + 2N_n^0 \geq 0. \quad (4.12)$$

Proof. The equality (4.12) follows by the sum of (4.10) and (4.11). ■

Theorem 4 *Let assumptions P1–P2 be satisfied and $\omega_+ > 0$. Let $N_A = \dim(\mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+)$ be the total number of isolated eigenvalues of A . Let $N_K = \dim(\mathcal{H}_K^- \oplus \mathcal{H}_K^+)$ be the total number of isolated eigenvalues of K . Assume that no embedded eigenvalues of the generalized eigenvalue problem (2.3) exist. Isolated eigenvalues of the generalized eigenvalue problem (2.3) satisfy the inequality:*

$$N_p^- + N_p^0 + N_p^+ + N_{c^+} \leq N_A + N_K. \quad (4.13)$$

Proof. We prove this theorem by contradiction. Let Π be a subspace in Π_κ spanned by eigenvectors of the generalized eigenvalue problem (2.3) which belong to N_p^- negative eigenvalues $\gamma < 0$, N_p^0 zero eigenvalues $\gamma = 0$, N_p^+ positive isolated eigenvalues $0 < \gamma < \omega_+\omega_-$, and N_{c^+} complex eigenvalues $\text{Im}(\gamma) > 0$. Let us assume that $N_p^- + N_p^0 + N_p^+ + N_{c^+} > N_A + N_K$. By Gram–Schmidt orthogonalization, there exist a vector h in Π such that $(h, f) = 0$ and $(h, g) = 0$, where $f \in \mathcal{H}_A^- \oplus \mathcal{H}_A^0 \oplus \mathcal{H}_A^+$ and $g \in \mathcal{H}_K^- \oplus \mathcal{H}_K^+$, such that $h \in \mathcal{H}_A^{\sigma_e(A)} \cap \mathcal{H}_K^{\sigma_e(K)}$. As a result,

$$(Ah, h) \geq \omega_+(h, h), \quad (Kh, h) \leq \omega_-^{-1}(h, h),$$

and

$$(Ah, h) \geq \omega_+\omega_-(Kh, h).$$

On the other hand, since $h \in \Pi$, we represent h by a linear combination of the eigenvectors in the corresponding subspaces of Π , such that

$$\begin{aligned} (Ah, h) &= \sum_{i,j} \alpha_i \bar{\alpha}_j (Ah_i, h_j) \\ &= \sum_{\gamma_i = \gamma_j < 0} \alpha_i \bar{\alpha}_j (Ah_i, h_j) + \sum_{\gamma_i = \gamma_j = 0} \alpha_i \bar{\alpha}_j (Ah_i, h_j) + \sum_{0 < \gamma_i = \gamma_j < \omega_+ \omega_-} \alpha_i \bar{\alpha}_j (Ah_i, h_j), \end{aligned}$$

where we have used Lemma 4.11 and Corollary 4.16. By Lemma 4.12, the non-zero values in (Ah_i, h_j) occur only for eigenvalues with odd algebraic multiplicity $n = 2k + 1$ for eigenvectors (Af_{k+1}, f_{k+1}) , where f_{k+1} is the generalized eigenvector in the basis for $\tilde{\mathcal{H}}_0$. Since all these cases are similar to the case of simple eigenvalues, we obtain

$$\begin{aligned} (Ah, h) &= \sum_{\gamma_j < 0} |\alpha_j|^2 (Ah_j, h_j) + \sum_{\gamma_j = 0} |\alpha_j|^2 (Ah_j, h_j) + \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Ah_j, h_j) \\ &= \sum_{\gamma_j < 0} \gamma_j |\alpha_j|^2 (Kh_j, h_j) + \sum_{0 < \gamma_j < \omega_+ \omega_-} \gamma_j |\alpha_j|^2 (Kh_j, h_j) \\ &< \omega_+ \omega_- \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Kh_j, h_j), \end{aligned}$$

where we have used the fact that $(Kh_j, h_j) \geq 0$ in Π . On the other hand,

$$\begin{aligned} (Kh, h) &= \sum_{i,j} \alpha_i \bar{\alpha}_j (Kh_i, h_j) \\ &= \sum_{\gamma_j < 0} |\alpha_j|^2 (Kh_j, h_j) + \sum_{\gamma_j = 0} |\alpha_j|^2 (Kh_j, h_j) + \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Kh_j, h_j) \\ &\geq \sum_{0 < \gamma_j < \omega_+ \omega_-} |\alpha_j|^2 (Kh_j, h_j). \end{aligned}$$

Therefore, $(Ah, h) < \omega_+ \omega_- (Kh, h)$, which is a contradiction. \blacksquare

Corollary 4.21 *Let $N_{\text{total}} = N_A + N_K$ be the total number of isolated eigenvalues of operators A and K . Let $N_{\text{isol}} = N_p^- + N_n^- + N_p^0 + N_n^0 + N_p^+ + N_n^+ + N_{c^+} + N_{c^-}$ be the total number of isolated eigenvalues of the generalized eigenvalue problem (2.3). Under the assumptions of Theorem 4, it is true that*

$$N_{\text{isol}} \leq N_{\text{total}} + \dim(\mathcal{H}_K^-). \quad (4.14)$$

Proof. The inequality (4.14) follows by the sum of (4.11) and (4.12). \blacksquare

Remark 4.22 If a simple embedded eigenvalue with the corresponding eigenvector f_1 is included into consideration according to Remark 4.19, then the left-hand-side of the inequality (4.13) is increased by the number of simple embedded eigenvalues with $[f_1, f_1] \leq 0$. For each embedded eigenvalue, the right-hand-side in the inequality (4.14) is reduced by two if $[f_1, f_1] \leq 0$ and it remains the same if $[f_1, f_1] > 0$.

5 Applications

We shall describe three applications of the general analysis which are related to recent studies of stability of solitons and vortices in the nonlinear Schrödinger equation and solitons in the Korteweg–de Vries equation.

5.1 Solitons of the scalar nonlinear Schrödinger equation

Consider a scalar nonlinear Schrödinger (NLS) equation,

$$i\psi_t = -\Delta\psi + F(|\psi|^2)\psi, \quad \Delta = \partial_{x_1x_1}^2 + \dots + \partial_{x_dx_d}^2, \quad (5.1)$$

where $(x, t) \in \mathbb{R}^d \times \mathbb{R}$ and $\psi \in \mathbb{C}$. For a suitable nonlinear function $F(|\psi|^2)$, where F is C^∞ and $F(0) = 0$, the NLS equation (5.1) possesses a solitary wave solution $\psi = \phi(x)e^{i\omega t}$, where $\omega > 0$, $\phi : \mathbb{R}^d \rightarrow \mathbb{R}$, and $\phi(x)$ is exponentially decaying C^∞ function. See [M93] for existence and uniqueness of ground state solutions to the NLS equation (5.1). Linearization of the NLS equation (5.1) with the ansatz,

$$\psi = \left(\phi(x) + [u(x) + iw(x)]e^{\lambda t} + [\bar{u}(x) + i\bar{w}(x)]e^{\bar{\lambda}t} \right) e^{i\omega t}, \quad (5.2)$$

where $\lambda \in \mathbb{C}$ and $(u, w) \in \mathbb{C}^2$, results in the linearized Hamiltonian problem (2.1), where L_\pm are Schrödinger operators,

$$L_+ = -\Delta + \omega + F(\phi^2) + 2\phi^2 F'(\phi^2), \quad (5.3)$$

$$L_- = -\Delta + \omega + F(\phi^2). \quad (5.4)$$

We note that L_\pm are unbounded operators and $\sigma_c(L_\pm) = [\omega, \infty)$ with $\omega_+ = \omega_- = \omega > 0$. The kernel of L_- includes at least one eigenvector $\phi(x)$ and the kernel of L_+ includes at least d eigenvectors $\partial_{x_j}\phi(x)$, $j = 1, \dots, d$. The Hilbert space \mathcal{X} is defined as $\mathcal{X} = H^1(\mathbb{R}^d, \mathbb{C})$ and the main assumptions P1-P2 are satisfied due to exponential decay of the potential functions $F(\phi^2)$ and $\phi^2 F'(\phi^2)$. Theorems 3 and 4 give precise count of eigenvalues of the stability problem $L_- L_+ u = -\lambda^2 u$, provided that the numbers $\dim(\mathcal{H}_K^-)$, $\dim(\mathcal{H}_{A+\delta K}^-)$, N_K and N_A can be computed from the count of isolated eigenvalues of L_\pm . We shall illustrate these computations with two examples.

Example 1. Let $\phi(x)$ be the ground state solution such that $\phi(x) > 0$ on $x \in \mathbb{R}^d$. By spectral theory, $\text{Ker}(L_-) = \{\phi(x)\}$ is one-dimensional and the subspace \mathcal{H}_K^- in (2.4) is empty, such that the corresponding Pontryagin space is Π_0 with $\kappa = 0$.

- By the equality (4.11), it follows immediately that $N_n^- = N_n^0 = N_n^+ = N_{c^+} = 0$ such that the spectrum of the problem (2.3) is real-valued and all eigenvalues are semi-simple.

- Since eigenvectors of $\text{Ker}(A)$ are in the positive subspace of K , the number of zero eigenvalues of the problem (2.3) is given by $N_p^0 = \dim(\mathcal{H}_A^0)$. If $\frac{d}{d\omega}\|\phi\|_{L^2}^2 = 0$, then $L_+\partial_\omega\phi(x) = -\phi(x)$, $\partial_\omega\phi(x) \in \mathcal{H}$, $\mathcal{P}\phi = 0$, and the eigenvector $\partial_\omega\phi(x) \in \text{Ker}(\mathcal{P}L_+\mathcal{P})$, such that $z_1 = 1$. If $z(L_+) = d$, then $z_0 = 0$ since $\partial_{x_j}\phi(x) \in \mathcal{H}$ for $j = 1, \dots, d$. In particular, $L_-\partial_{x_j}\phi(x) = -2\partial_{x_j}\phi(x)$ and $(K\partial_{x_j}\phi, \partial_{x_j}\phi) = \frac{1}{4}\|\phi\|_{L^2}^2 > 0$.
- By the equality (4.10), the number of negative eigenvalues of the problem (2.3) is given by $N_p^- = \dim(\mathcal{H}_{A+\delta K}^-)$. By Lemma 4.17 with $n = 1$ and $[f_1, f_1] > 0$, all zero eigenvalues of A become positive eigenvalues of $A + \delta K$ for $\delta > 0$. By Propositions 2.1 and 2.2, we have $\dim(\mathcal{H}_{A+\delta K}^-) = \dim(\mathcal{H}_A^-) = n(L_+) - n_0$. By Theorem 3.1 in [GSS90], $n_0 = 1$ if $\frac{d}{d\omega}\|\phi\|_{L^2}^2 > 0$ and $n_0 = 0$ otherwise.
- By the inequality (4.13), the number of positive eigenvalues of the problem (2.3) are bounded from above by $N_p^+ \leq \dim(\mathcal{H}_A^+) + \dim(\mathcal{H}_K^+)$. By Proposition 2.1, it is then $N_p^+ \leq p(L_+) + p(L_-) + n_0 + z_0 - z_1$.

Remark 5.1 Under the assumptions that $n(L_+) = 1$, $z(L_+) = d$, $p(L_+) = p(L_-) = 0$ and $\frac{d}{d\omega}\|\phi\|_{L^2}^2 < 0$, it follows from the above properties that $N_p^- = 1$, $N_p^0 = d$, and $N_p^+ = 0$. The assumptions $n(L_+) = 1$, $z(L_+) = d$ and $\frac{d}{d\omega}\|\phi\|_{L^2}^2 < 0$ are verified for the super-critical NLS equation with the power nonlinearity $F = |\psi|^p$ [S05]. Under the further assumptions $p(L_+) = p(L_-) = 0$, the statement $N_p^+ = 0$ is proved directly in Proposition 2.1.2 [P01] and Proposition 9.2 [KS05] for $d = 1$ and in Lemma 1.8 [S05] for $d = 3$. The same statement follows here by the count of eigenvalues from Theorem 4.

Remark 5.2 Systems of coupled NLS equations generalize the scalar NLS equation (5.1). Stability of vector solitons in the coupled NLS equations results in the same linearized Hamiltonian system (2.1) with the matrix Schrödinger operators L_\pm . General results for non-ground state solutions are obtained in [KKS04, P05] for $d = 1$ and in [CPV05] for $d = 3$. Multiple and embedded eigenvalues were either excluded from analysis by an assumption [P05, CPV05] or were treated implicitly [KKS04]. The present manuscript generalizes these results for an abstract case with a precise count of multiple and embedded eigenvalues.

Example 2. Let the scalar NLS equation (5.1) with $F = |\psi|^2$ be discretized with $\Delta \equiv \epsilon\Delta_{\text{disc}}$, where Δ_{disc} is the second-order discrete Laplacian and ϵ is a small parameter. We note that Δ_{disc} is a bounded operator and $\sigma_c(-\Delta_{\text{disc}}) \in [0, 4d]$. By the Lyapunov–Schmidt reduction method, the solution $\psi = \phi e^{i\omega t}$ with $\omega > 0$ bifurcates from the limit $\epsilon = 0$ with N non-zero lattice nodes to $\epsilon \neq 0$, such that $\frac{d}{d\omega}\|\phi\|_{L^2}^2 > 0$, $\ker(L_+) = \emptyset$, and $\ker(L_-) = \{\phi\}$ is one-dimensional for $0 < |\epsilon| < \epsilon_0$ with $\epsilon_0 > 0$ (see [PKF06a] for $d = 1$ and [PKF06b] for $d = 2$). By the equalities (4.10) and (4.11), we have

$$N_p^- + N_n^+ + N_{c^+} = n(L_+) - 1,$$

$$N_n^- + N_n^+ + N_{c+} = n(L_-),$$

where $n(L_+) = N$ and $n(L_-) \leq N - 1$ in the domain $0 < |\epsilon| < \epsilon_0$. When small positive eigenvalues of L_- are simple for $\epsilon \neq 0$, it is true that $N_n^+ = n(L_-)$, $N_n^- = N_{c+} = 0$, and $N_p^- = N - 1 - n(L_-)$ (see Corollary 3.5 in [PKF06a]⁴). It is clear that the Lyapunov–Schmidt reduction method gives a more precise information on numbers N_n^\pm , N_p^- , and N_{c+} , compared to the general equalities above. Similarly, by the inequality (4.13) and the counts above for $0 < |\epsilon| < \epsilon_0$, we have

$$N_p^+ \leq 2n(L_-) + \dim(\mathcal{H}_A^+) + \dim(\mathcal{H}_K^+).$$

When $0 < |\epsilon| < \epsilon_0$, the numbers N_p^+ and $\dim(\mathcal{H}_A^+)$ give precisely the numbers of edge bifurcations from the essential spectrum of $K^{-1}A$ and A respectively, while the number $\dim(\mathcal{H}_K^+)$ exceeds the number of edge bifurcations from the essential spectrum of K^{-1} by $N - 1 - n(L_-)$. When the solution ϕ is a ground state, we have $N = 1$ and $n(L_-) = 0$, such that the edge bifurcations in the spectrum of $K^{-1}A$ may only occur if there are edge bifurcations in the spectrum of A and K^{-1} for $\epsilon \neq 0$. When $N > 1$, the relation between edge bifurcations in the self-adjoint and non-self-adjoint problems become less direct.

Remark 5.3 The Lyapunov–Schmidt reduction method was also used for continuous coupled NLS equations with and without external potentials. See [KK04, KK06, PY05] for various results on the count of unstable eigenvalues in parameter continuations of the NLS equations.

5.2 Vortices of the scalar nonlinear Schrödinger equation

Consider the scalar two-dimensional NLS equation (5.1) with $d = 2$ in polar coordinates (r, θ) :

$$i\psi_t = -\Delta\psi + F(|\psi|^2)\psi, \quad \Delta = \partial_{rr}^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta}^2, \quad (5.5)$$

where $r > 0$ and $\theta \in [0, 2\pi]$. Assume that the NLS equation (5.5) possesses a charge- m vortex solution $\psi = \phi(r)e^{im\theta + i\omega t}$, where $\omega > 0$, $m \in \mathbb{N}$, $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$, and $\phi(r)$ is exponentially decaying C^∞ function for $r > 0$ with $\phi(0) = 0$. See [PW02] for existence results of charge- m vortices in the cubic-quintic NLS equation with $F = -|\psi|^2 + |\psi|^4$. Linearization of the NLS equation (5.5) with the anzats,

$$\psi = \left(\phi(r)e^{im\theta} + \varphi_+(r, \theta)e^{\lambda t} + \bar{\varphi}_-(r, \theta)e^{\bar{\lambda}t} \right) e^{i\omega t}, \quad (5.6)$$

where $\lambda \in \mathbb{C}$ and $(\varphi_+, \varphi_-) \in \mathbb{C}^2$, results in the stability problem,

$$\sigma_3 H \varphi = i\lambda \varphi, \quad (5.7)$$

⁴Corollary 3.5 in [PKF06a] is valid only when small positive eigenvalues of L_- are simple. It is shown in [PKF06b] that the case of multiple small positive eigenvalues of L_- leads to splitting of real eigenvalues N_p^- of the generalized eigenvalue problem (2.3) to complex eigenvalues N_{c+} beyond the leading-order Lyapunov–Schmidt reduction. The case of multiple small negative eigenvalues of L_- does not lead to this complication since the semi-simple purely imaginary eigenvalues N_n^+ do not split to complex eigenvalues N_{c+} .

where $\varphi = (\varphi_+, \varphi_-)^T$, $\sigma_3 = \text{diag}(1, -1)$, and

$$H = \begin{pmatrix} -\Delta + \omega + F(\phi^2) + \phi^2 F'(\phi^2) & \phi^2 F'(\phi^2) e^{2im\theta} \\ \phi^2 F'(\phi^2) e^{-2im\theta} & -\Delta + \omega + F(\phi^2) + \phi^2 F'(\phi^2) \end{pmatrix}.$$

Expand $\varphi(r, \theta)$ in the Fourier series

$$\varphi = \sum_{n \in \mathbb{Z}} \varphi^{(n)}(r) e^{in\theta}$$

and reduce the problem to a sequence of spectral problems for ODEs:

$$\sigma_3 H_n \varphi_n = i\lambda \varphi_n, \quad n \in \mathbb{Z}, \quad (5.8)$$

where $\varphi_n = (\varphi_+^{(n+m)}, \varphi_-^{(n-m)})^T$, and

$$H_n = \begin{pmatrix} -\partial_{rr}^2 - \frac{1}{r} \partial_r + \frac{(n+m)^2}{r^2} + \omega + F(\phi^2) + \phi^2 F'(\phi^2) & \phi^2 F'(\phi^2) \\ \phi^2 F'(\phi^2) & -\partial_{rr}^2 - \frac{1}{r} \partial_r + \frac{(n-m)^2}{r^2} + \omega + F(\phi^2) + \phi^2 F'(\phi^2) \end{pmatrix}.$$

When $n = 0$, the stability problem (5.8) transforms to the linearized Hamiltonian system (2.1), where L_{\pm} is given by (5.3)–(5.4) with $\Delta = \partial_{rr}^2 + \frac{1}{r} \partial_r - \frac{m^2}{r^2}$ and (u, w) are given by $u = \varphi_+^{(m)} + \varphi_-^{(-m)}$ and $w = -i(\varphi_+^{(m)} - \varphi_-^{(-m)})$. When $n \in \mathbb{N}$, the stability problem (5.8) transforms to the linearized Hamiltonian system (2.1) with $L_+ = H_n$ and $L_- = \sigma_3 H_n \sigma_3$, where

$$L_+ = L_- + 2\phi^2 F'(\phi^2) \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and (u, w) are given by $u = \varphi_n$ and $w = -i\sigma_3 \varphi_n$. When $-n \in \mathbb{N}$, the spectrum of the stability problem (5.8) can be obtained from that for $n \in \mathbb{N}$ by the correspondence $H_{-n} = \sigma_1 H_n \sigma_1$.

Let us introduce the weighted inner product for functions on $r \geq 0$:

$$(f, g) = \int_0^\infty f(r) g(r) r dr.$$

In all cases $n = 0$, $n \in \mathbb{N}$ and $-n \in \mathbb{N}$, L_{\pm} are unbounded self-adjoint differential operators and $\sigma_c(L_{\pm}) = [\omega, \infty)$, such that $\omega_+ = \omega_- = \omega > 0$. The kernel of H_n includes at least one eigenvector for $n = \pm 1$

$$\phi_{\pm 1} = \phi'(r) \mathbf{1} \mp \frac{m}{r} \phi(r) \sigma_3 \mathbf{1}, \quad \mathbf{1} = (1, 1)^T,$$

and at least one eigenvector for $n = 0$: $\phi_0 = \phi(r) \sigma_3 \mathbf{1}$. The Hilbert space \mathcal{X} associated with the weighted inner product is defined as $\mathcal{X} = H^1(\mathbb{R}_+, \mathbb{C})$ for $n = 0$ and $\mathcal{X} = H^1(\mathbb{R}_+, \mathbb{C}^2)$ for $\pm n \in \mathbb{N}$. In all cases, the main assumptions P1-P2 are satisfied due to exponential decay of the potential functions $F(\phi^2)$ and $\phi^2 F'(\phi^2)$.

The case $n = 0$ is the same as for solitons (see Section 5.1). We shall hence consider adjustments in the count of eigenvalues in the case $\pm n \in \mathbb{N}$, when the stability problem (5.8) is rewritten in the form,

$$\begin{cases} \sigma_3 H_n \varphi_n = i\lambda \varphi_n \\ \sigma_3 H_{-n} \varphi_{-n} = i\lambda \varphi_{-n} \end{cases} \quad n \in \mathbb{N}. \quad (5.9)$$

Let L_+ be a diagonal composition of H_n and H_{-n} and L_- be a diagonal composition of $\sigma_3 H_n \sigma_3$ and $\sigma_3 H_{-n} \sigma_3$.

Lemma 5.4 *Let λ be an eigenvalue of the stability problem (5.9) with the eigenvector $(\varphi_n, \mathbf{0})$. Then there exists another eigenvalue $-\lambda$ with the linearly independent eigenvector $(\mathbf{0}, \sigma_1 \varphi_n)$. If $\text{Re}(\lambda) > 0$, there exist two more eigenvalues $\bar{\lambda}, -\bar{\lambda}$ with the linearly independent eigenvectors $(\mathbf{0}, \sigma_1 \bar{\varphi}_n)$, $(\bar{\varphi}_n, \mathbf{0})$.*

Proof. We note that $\sigma_1 \sigma_3 = -\sigma_3 \sigma_1$ and $\sigma_1^2 = \sigma_3^2 = \sigma_0$, where $\sigma_0 = \text{diag}(1, 1)$. Therefore, each eigenvalue λ of H_n with the eigenvector φ_n generates eigenvalue $-\lambda$ of H_{-n} with the eigenvector $\varphi_{-n} = \sigma_1 \varphi_n$. When $\text{Re}(\lambda) \neq 0$, each eigenvalue λ of H_n generates also eigenvalue $-\bar{\lambda}$ of H_n with the eigenvector $\bar{\varphi}_n$ and eigenvalue $\bar{\lambda}$ of H_{-n} with the eigenvector $\varphi_{-n} = \sigma_1 \bar{\varphi}_n$. ■

Theorem 5 *Let N_{real} be the number of real eigenvalues in the stability problem (5.9) with $\text{Re}(\lambda) > 0$, N_{comp} be the number of complex eigenvalues with $\text{Re}(\lambda) > 0$ and $\text{Im}(\lambda) > 0$, N_{imag}^- be the number of purely imaginary eigenvalues with $\text{Im}(\lambda) > 0$ and $(\varphi_n, H_n \varphi_n) \leq 0$, and N_{zero}^- is the algebraic multiplicity of the zero eigenvalue of $\sigma_3 H_n \varphi_n = i\lambda \varphi_n$ counted in Remark 4.19. Assume that all embedded eigenvalues are simple. Then,*

$$\frac{1}{2}N_{\text{real}} + N_{\text{comp}} = n(H_n) - N_{\text{zero}}^- - N_{\text{imag}}^-, \quad (5.10)$$

where N_{real} is even. Moreover, $n_0 = n_-$, where n_0 and n_- are defined by Propositions 2.1 and 2.2.

Proof. By Lemma 5.4, a pair of real eigenvalues of $\sigma_3 H_n \varphi_n = i\lambda \varphi_n$ corresponds to two linearly independent eigenvectors φ_n and $\bar{\varphi}_n$. Because $(H_n \varphi_n, \varphi_n)$ is real-valued and hence zero for $\lambda \in \mathbb{R}$, we have

$$(H_n(\varphi_n \pm \bar{\varphi}_n), (\varphi_n \pm \bar{\varphi}_n)) = \pm 2\text{Re}(H_n \varphi_n, \bar{\varphi}_n).$$

By counting multiplicities of the real negative and complex eigenvalues of the problem (2.3) associated to the stability problem (5.9), we have $N_n^- = N_p^- = N_{\text{real}}$ and $N_{c^+} = 2N_{\text{comp}}$. By Lemma 5.4, a pair of purely imaginary and zero eigenvalues of the stability problem (5.9) corresponds to two linearly independent eigenvectors $(\varphi_n, \mathbf{0})$ and $(\mathbf{0}, \varphi_{-n})$, where $\varphi_{-n} = \sigma_1 \varphi_n$ and $(H_{-n} \varphi_{-n}, \varphi_{-n}) = (H_n \varphi_n, \varphi_n)$. By counting multiplicities of the real positive and zero eigenvalues of the problem (2.3) associated to the stability problem (5.9), we have $N_n^0 = 2N_{\text{zero}}^-$ and $N_n^+ = 2N_{\text{imag}}^-$. Since the spectra of H_n , $\sigma_1 H_n \sigma_1$, and $\sigma_3 H_n \sigma_3$ coincide, we have $n(L_+) = n(L_-) = 2n(H_n)$. As a result, the equality (5.10) follows by the equality (4.11) of Theorem 3. By Lemma 5.4, the multiplicity of N_{real} is even in the stability problem (5.9). The other equality (4.10) of Theorem 3 recovers the same answer provided that $\dim(\mathcal{H}_{A+\delta K}^-) = \dim(\mathcal{H}_A^-) + n_- = n(L_+) - n_0 + n_- = n(L_+)$, such that $n_0 = n_-$. ■

Example 3. Let $\phi(r)$ be the fundamental charge- m vortex solution such that $\phi(r) > 0$ for $r > 0$ and $\phi(0) = 0$. By spectral theory, $\text{Ker}(H_0)$ is one-dimensional with the eigenvector ϕ_0 . The analysis of $n = 0$

is similar to Example 1. In the case $n \in \mathbb{N}$, we shall assume that $\text{Ker}(H_1) = \{\phi_1\}$ and $\text{Ker}(H_n) = \emptyset$ for $n \geq 2$.

- Since $(\sigma_3 \phi_1, \phi_1) = 0$ and $\text{Ker}(\sigma_3 H_1 \sigma_3) = \{\sigma_3 \phi_1\}$, then $\phi_1 \in \mathcal{H}$, such that $z_0 = 0$.
- By direct computation, $(\sigma_3 H_1 \sigma_3)^{-1} \phi_1 = -\frac{1}{2} r \phi(r) \mathbf{1}$ and

$$((\sigma_3 H_1 \sigma_3)^{-1} \phi_1, \phi_1) = \int_0^\infty r \phi^2(r) dr > 0.$$

By Lemma 4.17, we have $N_n^0 = n_- = 0$ for $n = 1$, as well as for $n \geq 2$. By Proposition 2.1, we have $A(0) < 0$ such that $n_0 = z_1 = 0$ for all $n \in \mathbb{N}$.

- By Theorem 5, we have

$$N_{\text{real}} + 2N_{\text{comp}} = 2n(H_n) - 2N_{\text{imag}}^-. \quad (5.11)$$

If all purely imaginary eigenvalues are semi-simple and isolated, N_{imag}^- gives the total number of eigenvalues in the stability problem (5.9) with $\text{Re}(\lambda) = 0$, $\text{Im}(\lambda) > 0$, and negative Krein signature $(H_n \varphi_n, \varphi_n) < 0$.

Remark 5.5 Stability of vortices was considered numerically in [PW02], where Lemma 5.4 was also obtained. The closure relation (5.11) was also discussed in [KKS04] in a more general context. Detailed comparison of numerical results and the closure relation (5.11) can be found in [K05]. Vortices in the discretized scalar NLS equation were considered with the Lyapunov–Schmidt reduction method in [PKF06b]. Although the reduced eigenvalue problems were found in a much more complicated form compared to the reduced eigenvalue problem for solitons, the relation (5.11) was confirmed in all particular vortex configurations considered in [PKF06b].

Remark 5.6 Since N_{real} is even, splitting of simple complex eigenvalues N_{comp} into double real eigenvalues N_{real} is prohibited by the closure relation (5.11). Simple complex eigenvalues may either coalesce into a double real eigenvalue that persists or reappear again as simple complex eigenvalues.

5.3 Solitons of the fifth-order Korteweg–De Vries equation

Consider a general fifth-order KdV equation,

$$v_t = a_1 v_x - a_2 v_{xxx} + a_3 v_{xxxxx} + 3b_1 v v_x - b_2 (v v_{xxx} + 2v_x v_{xx}) + 6b_3 v^2 v_x, \quad (5.12)$$

where (a_1, a_2, a_3) and (b_1, b_2, b_3) are real-valued coefficients for linear and nonlinear terms, respectively. Without loss of generality, we assume that $a_3 > 0$ and

$$c_{\text{wave}}(k) = a_1 + a_2 k^2 + a_3 k^4 \geq 0, \quad \forall k \in \mathbb{R}. \quad (5.13)$$

For suitable values of parameters, there exists a traveling wave solution $v(x, t) = \phi(x - ct)$, where $c > 0$ and $\phi : \mathbb{R} \mapsto \mathbb{R}$, such that the function $\phi(x)$ is even and exponentially decaying as $|x| \rightarrow \infty$. Existence of traveling waves was established in [Z87, HS88, AT92] for $b_2 = b_3 = 0$, in [CG97] for $b_3 = 0$, in [K97] for $b_1 = -b_2 = b_3 = 1$, and in [L99] for $b_3 = 0$ or $b_1 = b_2 = 0$. Linearization of the fifth-order KdV equation (5.12) with the ansatz

$$v(x, t) = \phi(x - ct) + w(x - ct)e^{\lambda t}$$

results in the stability problem

$$\partial_x L_- w = \lambda w, \quad (5.14)$$

where L_- is an unbounded fourth-order operator,

$$L_- = a_3 \frac{d^4}{dx^4} - a_2 \frac{d^2}{dx^2} + a_1 + c + 3b_1 \phi(x) - b_2 \frac{d}{dx} \phi(x) \frac{d}{dx} - b_2 \phi''(x) + 6b_3 \phi^2(x). \quad (5.15)$$

With the account of the condition (5.13), the continuous spectrum of L_- is located for $\sigma_c(L_-) \in [c, \infty)$, such that $\omega_- = c > 0$. The kernel of L_- includes at least one eigenvector $\phi'(x)$. Since the image of L_- is in $L^2(\mathbb{R})$, the eigenfunction $w(x) \in L^1(\mathbb{R})$ for $\lambda \neq 0$ satisfies the constraint:

$$(1, w) = \int_{\mathbb{R}} w(x) dx = 0. \quad (5.16)$$

Let $w = u'(x)$, where $u(x) \rightarrow 0$ as $|x| \rightarrow \infty$ and define $L_+ = -\partial_x L_- \partial_x$. The continuous spectrum of L_+ is located for $\sigma_c(L_+) \in [0, \infty)$, such that $\omega_+ = 0$. The kernel of L_+ includes at least one eigenvector $\phi(x)$.

Let the Hilbert space \mathcal{X} be defined as $\mathcal{X} = H^3(\mathbb{R}, \mathbb{C})$. The main assumptions P1-P2 for L_- and L_+ are satisfied due to exponential decay of the potential function $\phi(x)$. The stability problem (5.14) is equivalent to the linearized Hamiltonian system (2.1). The operator L_- defines the constrained subspace \mathcal{H} and hence the Pontryagin space Π_κ with $\kappa = n(L_-)$. The operator L_+ has the embedded kernel to the endpoint of the essential spectrum of L_+ . This introduces a technical complication in computations of the inverse of L_+ [KP05], which we avoid here with the use of the shifted generalized eigenvalue problem (2.7) with $\delta > 0$. It is easy to estimate that

$$\omega_{A+\delta K} = \inf_{k \in \mathbb{R}} \left[k^2(c + c_{\text{wave}}(k)) + \frac{\delta}{c + c_{\text{wave}}(k)} \right] \geq \frac{\delta}{c} > 0,$$

where $\omega_{A+\delta K}$ is defined below the decomposition (2.8). Theorem 3 can be applied after appropriate adjustments in the count of isolated and embedded eigenvalues in the stability problem (5.14). Since $\omega_+ = 0$, Theorem 4 is not applicable to the fifth-order KdV equation (5.12). Because the continuous spectrum of $\partial_x L_-$ is on $\lambda \in i\mathbb{R}$, all real and complex eigenvalues are isolated and all purely imaginary eigenvalues including the zero eigenvalue are embedded.

Lemma 5.7 *Let λ_j be a real eigenvalue of the stability problem (5.14) with the real-valued eigenvector $w_j(x)$, such that $\operatorname{Re}(\lambda_j) > 0$ and $\operatorname{Im}(\lambda_j) = 0$. Then there exists another eigenvalue $-\lambda_j$ in the problem (5.14) with the linearly independent eigenvector $w_j(-x)$. The linear combinations $w_j^\pm(x) = w_j(x) \pm w_j(-x)$ are orthogonal with respect to the operator L_- ,*

$$\left(L_- w_j^\pm, w_j^\pm\right) = \pm 2\rho_j, \quad \left(L_- w_j^\mp, w_j^\pm\right) = 0, \quad (5.17)$$

where $\rho_j = (L_- w_j(-x), w_j(x))$.

Proof. Since $\phi(-x) = \phi(x)$, the self-adjoint operator L_- is invariant with respect to the transformation $x \mapsto -x$. The functions $w_j(x)$ and $w_j(-x)$ are linearly independent since $w_j(x)$ has both symmetric and anti-symmetric parts provided that $\lambda_j \neq 0$. Under the same constraint,

$$(L_- w_j(\pm x), w_j(\pm x)) = \pm \lambda_j^{-1} (L_- w_j(\pm x), \partial_x L_- w_j(\pm x)) = 0,$$

and the orthogonality relations (5.17) hold by direct computations. ■

Corollary 5.8 *Let λ_j be a complex eigenvalue of the stability problem (5.14) with the complex-valued eigenvector $w_j(x)$, such that $\operatorname{Re}(\lambda_j) > 0$ and $\operatorname{Im}(\lambda_j) > 0$. Then there exist eigenvalues $\bar{\lambda}_j$, $-\lambda_j$, and $-\bar{\lambda}_j$ in the problem (5.14) with the linearly independent eigenvectors $\bar{w}_j(x)$, $w_j(-x)$, and $\bar{w}_j(-x)$, respectively.*

Lemma 5.9 *Let λ_j be an embedded eigenvalue of the stability problem (5.14) with the complex-valued eigenvector $w_j(x)$, such that $\operatorname{Re}(\lambda_j) = 0$ and $\operatorname{Im}(\lambda_j) > 0$. Then there exists another eigenvalue $-\lambda_j = \bar{\lambda}_j$ in the problem (5.14) with the linearly independent eigenvector $w_j(-x) = \bar{w}_j(x)$. The linear combinations $w_j^\pm(x) = w_j(x) \pm \bar{w}_j(x)$ are orthogonal with respect to the operator L_- ,*

$$\left(L_- w_j^\pm, w_j^\pm\right) = 2\rho_j, \quad \left(L_- w_j^\mp, w_j^\pm\right) = 0, \quad (5.18)$$

where $\rho_j = \operatorname{Re}(L_- w_j(x), w_j(x))$.

Proof. Since operator L_- is real-valued, the eigenvector $w_j(x)$ of the problem (5.14) with $\operatorname{Im}(\lambda_j) > 0$ has both real and imaginary parts, which are linearly independent. Under the constraint $\lambda_j \neq 0$,

$$(L_- w_j, \bar{w}_j) = \lambda_j^{-1} (L_- w_j, \partial_x L_- \bar{w}_j) = 0,$$

and the orthogonality equations (5.18) follow by direct computations. ■

Theorem 6 *Let N_{real} be the number of real eigenvalues of the stability problem (5.14) with $\operatorname{Re}(\lambda) > 0$, N_{comp} be the number of complex eigenvalues with $\operatorname{Re}(\lambda) > 0$ and $\operatorname{Im}(\lambda) > 0$, and N_{imag}^- be the number*

of imaginary eigenvalues with $\text{Im}(\lambda) > 0$ and $\rho_j \leq 0$ in (5.18). Assume that all embedded (zero and imaginary) eigenvalues of the stability problem (5.14) are simple, such that $\text{Ker}(L_+) = \{\phi\} \in \mathcal{H}$. Then,

$$N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^- = n(L_-) - N_{\text{zero}}^-, \quad (5.19)$$

where $N_{\text{zero}}^- = 1$ if $(L_-^{-1}\phi, \phi) \leq 0$ and $N_{\text{zero}}^- = 0$ otherwise.

Proof. Each isolated and embedded eigenvalue $\gamma_j = -\lambda_j^2$ of the generalized eigenvalue problem (2.3) is at least double with two linearly independent eigenvectors $u_j^\pm(x)$, such that $w_j^\pm = \partial_x u_j^\pm$. By Lemma 5.7 and Corollary 5.8, the dimension of non-positive invariant subspace of Π_κ for isolated (real and complex) eigenvalues coincide with the algebraic multiplicities of isolated eigenvalues, such that $N_n^- = N_p^- = N_{\text{real}}$ and $N_{c+} = 2N_{\text{comp}}$. By convention in Remark 4.14, all embedded (zero and purely imaginary) eigenvalues are assumed to be simple. By Lemma 5.9 and the relation for eigenvectors of the stability problem (5.14),

$$(L_+ u, u) = (L_- u', u') = (L_- w, w), \quad (5.20)$$

we have $N_n^+ = 2N_{\text{imag}}^-$ and $N_n^0 = N_{\text{zero}}^0$. The count (5.19) follows by equality (4.11) of Theorem 3. ■

Remark 5.10 Theorem 6 can be generalized to any KdV-type evolution equation, when the linearization operator L_- is invariant with respect to the transformation $x \mapsto -x$. When $N_{\text{imag}}^- = N_n^0 = 0$, the relation (5.19) extends the Morse index theory from gradient dynamical systems to the KdV-type Hamiltonian systems. For gradient dynamical systems, all negative eigenvalues of L_- are related to real unstable eigenvalues of the stability problem. For the KdV-type Hamiltonian system, negative eigenvalues of L_- may generate both real and complex unstable eigenvalues in the problem (5.14).

Example 4. Let $\phi(x)$ be a single-pulse solution on $x \in \mathbb{R}$ (which does not have to be positive). Assume that the operator L_- has a one-dimensional kernel in \mathcal{X} with the eigenvector $\phi'(x)$. Then the kernel of L_+ is one-dimensional in \mathcal{X} with the eigenvector $\phi(x)$, such that $z(L_+) = 1$.

- Since $(\phi, \phi') = 0$, then $\text{Ker}(L_+) \in \mathcal{H}$ and $z_0 = 0$. Since $L_- \partial_c \phi(x) = -\phi(x)$, we have $z_1 = 1$ if $\frac{d}{dc} \|\phi\|_{L^2}^2 = 0$. Since the latter case violates the assumption that the embedded kernel is simple in \mathcal{H} , we shall consider the case $\frac{d}{dc} \|\phi\|_{L^2}^2 \neq 0$, such that $z_1 = 0$.
- Since $(K\phi, \phi) = -(\partial_c \phi, \phi) = -\frac{1}{2} \frac{d}{dc} \|\phi\|_{L^2}^2$, we have $N_{\text{zero}}^- = 1$ if $\frac{d}{dc} \|\phi\|_{L^2}^2 > 0$ and $N_{\text{zero}}^- = 0$ if $\frac{d}{dc} \|\phi\|_{L^2}^2 < 0$. By Lemma 4.17 and Remark 4.18, $\dim(\mathcal{H}_{A+\delta K}^-) = \dim(\mathcal{H}_A^-) + N_{\text{zero}}^-$. By the relation (4.10) of Theorem 3 with $N_p^- = N_{\text{real}}$, we have

$$N_{\text{real}} + 2N_{\text{comp}} + 2N_{\text{imag}}^- = \dim(\mathcal{H}_{A+\delta K}^-) - N_{\text{zero}}^- = \dim(\mathcal{H}_A^-). \quad (5.21)$$

- Recall that $\dim(\mathcal{H}_A^-) = n(L_+) - n_0$ in Proposition 2.1 (extended to the case $\omega_+ = 0$). By the Sylvester Inertia Law Theorem and the relation (5.20), we have $n(L_+) = n(L_-)$ in \mathcal{X} . Since $A(0) = -(L_+^{-1}\phi', \phi') = -(L_-^{-1}\phi, \phi)$, we have $n_0 = N_{\text{zero}}^-$ in \mathcal{H} . As a result, the relation (5.21) recovers the same equality (5.19).

Remark 5.11 General stability-instability results for the traveling waves of the KdV-type equations were obtained in [BSS87, SS90] when $n(L_-) = 1$. In this case, $N_{\text{comp}} = N_{\text{imag}}^- = 0$ and $N_{\text{real}} = 1 - N_{\text{zero}}^-$, such that stability follows from $\frac{d}{dc}\|\phi\|_{L^2}^2 > 0$ and instability follows from $\frac{d}{dc}\|\phi\|_{L^2}^2 < 0$. By a different method, Lyapunov stability of positive traveling waves $\phi(x)$ was considered in [W87]. Specific studies of stability for the fifth-order KdV equation (5.12) were reported in [IS92, DK99] with the energy-momentum methods. Extension of the stability-instability theorems of [BSS87, W87] with no assumption on a simple negative eigenvalue of L_- was developed in [L99, P03] with a variational method. The variational theory is limited however to the case of homogeneous nonlinearities, e.g. $b_3 = 0$ or $b_1 = b_2 = 0$. Our treatment of stability in the fifth-order KdV equation (5.12) is completely new and it exploits a similarity between stability problems of KdV and NLS solitons. The pioneer application of the new theory to stability of N-solitons in the KdV hierarchy is reported in [KP05]. Further progress on the same topic will appear soon in [BCD06, S06].

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